A functional analytic approach to Stirling numbers of the second kind

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Abstract

Using holomorphic functional calculus we show that the Stirling numbers of the second kind \( S(j, i) \) can be obtained as entries of the \( i \)-th power of a certain nilpotent matrix. This yields a recurrence relation and a matrix representation of the Bell numbers.

Lemma 1. Let \( p \in \mathbb{N}_0 \) and \( \mathbb{B}^{(p)} \in \text{Mat}(p+1, p+1; \mathbb{R}) \) be the unipotent (lower) triangular matrix with entries

\[
\left( \mathbb{B}^{(p)} \right)_{i,j} = \begin{cases} \binom{i}{j}, & \text{if } 0 \leq j \leq i \leq p, \\ 0, & \text{otherwise}. \end{cases}
\]

Then for all \( z \in \mathbb{C} \) the matrix \( (\mathbb{B}^{(p)})^z \) defined via holomorphic functional calculus is given by

\[
(\mathbb{B}^{(p)})^z}_{i,j} = \begin{cases} z^{i-j} \binom{i}{j}, & \text{if } 0 \leq j \leq i \leq p, \\ 0, & \text{otherwise}. \end{cases}
\]

Proof. Consider \( f_z(u) = u^z \) whose derivatives are \( \partial_u^i f_z(u) = \prod_{k=0}^{i-1} (z-k) u^{z-i} \). The holomorphic functional calculus (cf. [3, VII.1]) shows that for any unipotent matrix \( \mathbb{B} \in \text{Gl}(p+1; \mathbb{C}) \) one has

\[
\mathbb{B}^z = \sum_{i=0}^{p} \frac{\partial_u^i f_z(u)}{i!}(\mathbb{B} - 1)^i = 1 + \sum_{i=1}^{p} \prod_{k=0}^{i-1} \frac{(z-k)}{i!} (\mathbb{B} - 1)^i.
\]

Hence for any \( 0 \leq i, j \leq p \) the coefficients \( (\mathbb{B}^z)_{i,j} \) polynomially depend on \( z \) (and polynomially on \( \mathbb{B} \), too). Thus it suffices to verify that (1) holds for all \( z \in \mathbb{N}_0 \). For \( i \geq j \) one has

\[
(\mathbb{B}^{(p)})^{n+1}_{i,j} = \sum_{k=j}^{i} n^{i-k} \binom{i}{k} \binom{k}{j} = \binom{i}{j} \sum_{l=0}^{i-j} n^{i-j-l} \binom{i-j}{l} = \binom{i}{j} (n+1)^{i-j}
\]

and hence induction concludes the proof. \( \square \)

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For \( z = -1 \) Lemma 1 yields a well-known Gould type inversion formula, cf. [2, 1.7]. We found Lemma 1 in the context of transfer operators for certain classical spin systems, [4, 3.4], where we used that \( B(p) \) is the representing matrix of the shift \( (\tau f)(n) = f(n + 1) \) on the space of polynomials of degree less or equal \( p \) with respect to the basis \( 1, z, \ldots, z^p \). The difference operator \( \Delta = \tau - 1 \), \( (\Delta f)(n) = f(n + 1) - f(n) \) is strongly related to the Stirling numbers of the second kind \( S(j, i) \). Recall that

\[
S(j, i) := S^{(i)}_j := \frac{1}{i!} \sum_{k=0}^{i} \binom{i}{k} (-1)^k (i-k)^j,
\]

which counts the number of ways of partitioning a set of \( j \) elements into \( i \) non-empty subsets, cf. [1, 24.1.4] or [5, 1.4 (24a)].

**Theorem 2.** For \( i, j, p \in \mathbb{N}_0 \) with \( j \leq p \) one has

\[
S(j, i) = \frac{1}{i!} (B(p) - 1)^i_{j,0}.
\]

**Proof.** For any \( j, p \in \mathbb{N}_0 \) with \( j \leq p \) we expand

\[
(B(p) - 1)^i_{j,0} = \sum_{n=0}^{i} \binom{i}{n} (-1)^{i-n} n^j = \sum_{k=0}^{i} \binom{i}{k} (-1)^k (i-k)^j
\]

via the binomial formula and (1).

Considering a certain interpolation problem yields some of the well-known identities for the Stirling numbers of the second kind.

**Corollary 3.** For all \( z \in \mathbb{C} \) one has

\[
z^j = \sum_{i=0}^{j} S(j, i) \prod_{k=0}^{i-1} (z - k).
\]

In particular, \( S(j, 1) = S(j, j) = 1 \), \( S(j, 0) = 0 \), and \( S(j, i) = 0 \) for all \( i > j \).

**Proof.** Let \( p \geq j \). Equation (2) and Theorem 2 yield

\[
(3) \quad z^j = \left((B(p)^{z})\right)_{j,0} = \sum_{i=1}^{p} \prod_{k=0}^{i-1} (z - k) \left((B(p) - 1)^i_{j,0}\right) = \sum_{i=1}^{p} \prod_{k=0}^{i-1} (z - k) S(j, i).
\]

Since \( p_i : z \mapsto (z)_i = \prod_{k=0}^{i-1} (z - k) \) is a normed polynomial of degree \( i \), there is a unique solution of the interpolation problem

\[
z^j = \sum_{i=0}^{p} c_{j,i} p_i(z).
\]

By (3) and Lemma 1 this solution is given by the \( S(j, i) \). By the above interpretation as an interpolation problem one directly obtains the special values.

In other words, \( S(i, j)_{i,j=0,\ldots,p} \) is the representing matrix of the base change \( \{x_j \mid j = 0, \ldots, p\} \mapsto \{x^i \mid i = 0, \ldots, p\} \). The matrix representation from Theorem 2 yields the following recurrence relation.
Corollary 4.

\[ S(j, i + 1) = \frac{1}{(i + 1)!} \sum_{k=1}^{j-1} \binom{j}{k} S(k, i). \]

Proof. Set \( G := B(p) - 1 \) for some \( p \geq i, j \). Then by Theorem 2

\[ (i + 1)! S(j, i + 1) = (GG^i)_{i,0} = \sum_{k=0}^{p} G_{j,k} S(k, i) = \sum_{k=0}^{j-1} \binom{j}{k} S(k, i). \]

The last expression is equal to \( \sum_{k=1}^{j-1} \binom{j}{k} S(k, i) \) by Corollary 3.

The number of ways a set of \( j \) elements can be partitioned into non-empty subsets is called the Bell number \( B_j \). Clearly, \( B_j = \sum_{i=0}^{j} S(j, i) \).

Corollary 5. For any \( p \geq j \) one has

\[ B_j = (e^{B(p) - 1})_{j,0}. \]

Proof. We use that \( G := B(p) - 1 \in \text{Mat}(p+1, p+1; \mathbb{C}) \) is a strict lower triangular matrix. By Theorem 2 and Neumann’s series one has

\[ B_j = \sum_{i=0}^{j} S(j, i) = \left( \sum_{i=0}^{j} G_i \right)_{j,0} = \left( \sum_{i=0}^{\infty} G_i \right)_{j,0} = (e^G)_{j,0}. \]

Question: Can one use Cor. 5 in order to compute the Bell numbers efficiently? or: to obtain a (non-trivial) bound?

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